Int J Theor Phys (2009) 48: 913–923
DOI 10.1007/s10773-008-9864-1

The Quantum Reduced Action in Higher Dimensions

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Received: 18 July 2008 / Accepted: 26 September 2008 / Published online: 8 October 2008

Abstract The solution with respect to the reduced action of the one-dimensional stationary quantum Hamilton-Jacobi equation is well known in the literature. The extension to higher dimensions in the separated variable case was proposed in contradictory formulations. In this paper we provide new insights into the construction of the reduced action. In particular, contrary to the classical mechanics case, we analytically show that the reduced action constructed as a sum of one variable functions does not contain a complete information about the quantum motion. In the same context, we also make some observations about recent results concerning quantum trajectories. Finally, we will examine the conditions in which microstates appear even in the case where the wave function is complex.

Keywords Reduced action · Quantum stationary Hamilton-Jacobi equation · Separated variables · Trajectories · Microstates

1 Introduction

In the context of quantum trajectories, the notion of the action was first introduced by de Broglie [1] and Bohm [2, 3] by writing the wave function in the well-known form

$$\Psi = R \exp\left(i \frac{S}{\hbar}\right), \tag{1}$$

where R and S are real functions. The substitution of this expression in the time-dependent Schrödinger equation leads to the two following partial derivative equations

$$\frac{1}{2m}\left(\vec{\nabla} S\right)^2 - \frac{\hbar^2}{2m}\frac{\Delta R}{R} + V = -\frac{\partial S}{\partial t}, \tag{2}$$

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$$\frac{\partial R^2}{\partial t} + \vec{\nabla} \cdot (R^2 \vec{\nabla} S) = 0, \tag{3}$$

V being an external potential. Relation (3) is the continuity equation. Relation (2) reminds us of the Hamilton-Jacobi equation with an additional term

$$U = -\frac{\hbar^2}{2m} \frac{\Delta R}{R}, \tag{4}$$

called the quantum potential and to which it is attributed the quantum effects. The function S is identified as the quantum action and relation (2) is then called the quantum Hamilton-Jacobi equation (QHJE). In the stationary case, we have

$$S(x, y, z, t) = S_0(x, y, z) - Et, \tag{5}$$

and

$$\Psi(x, y, z, t) = \exp\left(-\frac{i}{\hbar} Et\right) \phi(x, y, z), \tag{6}$$

E being the energy and S_0 the quantum reduced action. By considering a system described by a wave function such that $\phi(x, y, z)$ is real up to a constant phase factor, relations (1), (5) and (6) show clearly that S_0 is constant. This feature is unsatisfactory. In the context of the equivalence postulate of quantum mechanics [4–7] from which the Schrödinger equation (SE) was reproduced, this difficulty was surmounted by showing that the wave function takes the form

$$\phi(x, y, z) = R\left[\alpha \exp\left(i \frac{S_0}{\hbar}\right) + \beta \exp\left(-i \frac{S_0}{\hbar}\right)\right], \tag{7}$$

where α and β are complex constants. This bipolar form was also used by Poirier [8] in order to reconcile semiclassical and Bohmian Mechanics. The new functions R and S_0 satisfy the equations

$$\frac{1}{2m}\left(\vec{\nabla} S_0\right)^2 - \frac{\hbar^2}{2m} \frac{\Delta R}{R} + V(x, y, z) = E, \tag{8}$$

$$\vec{\nabla} \cdot (R^2 \vec{\nabla} S_0) = 0, \tag{9}$$

which are fundamentally different from the stationary version of (2) and (3) since in the two cases the couple (R, S_0) is linked in different manners to the wave function. To perceive this difference, one can realize the form (7) guarantees that S_0 is never constant even in the case where the wave function is real, up to a constant phase factor. These results were reproduced in [9] by appealing to the probability current.

In one dimension, (8) and (9) turn out to be

$$\begin{aligned} &\frac{1}{2m}\left(\frac{\partial S_0}{\partial x}\right)^2 + V(x) - E \\ &\quad = \frac{\hbar^2}{4m}\left[\frac{3}{2}\left(\frac{\partial S_0}{\partial x}\right)^{-2}\left(\frac{\partial^2 S_0}{\partial x^2}\right)^2 - \left(\frac{\partial S_0}{\partial x}\right)^{-1}\left(\frac{\partial^3 S_0}{\partial x^3}\right)\right]. \end{aligned} \tag{10}$$

The solution of this equation for an arbitrary potential $V(x)$ has been investigated in [10, 11] and it is written as

$$S_0 = \hbar \arctan\left(\frac{b(\phi_1/\phi_2) + c/2}{(ab - c^2/4)^{1/2}}\right) + K, \tag{11}$$

where (a, b, c, K) is a set of real constants such that $a > 0$, $b > 0$ and $ab > c^2/4$. The couple (ϕ_1, ϕ_2) is a set of real independent solutions of the associated SE. The above expression of S_0 solves (10) if the Wronskian $W = \phi_1\phi_2' - \phi_1'\phi_2$ is scaled so that $W^2 \equiv 2m/[\hbar^2(ab - c^2/4)]$. As (10) is a second order differential equation with respect to $\partial S_0/\partial x$, the expression of S_0 must contain two integration constants on top of the additive one. This is the reason for which the three parameters (a, b, c) are not independent since they are linked to W for a given choice of the couple (ϕ_1, ϕ_2). Therefore, it is possible to eliminate one of them. That's what is done in [9] where the solution of (10) is written as

$$S_0 = \hbar \arctan\left(\frac{\phi_1 + \nu\phi_2}{\mu\phi_1 + \phi_2}\right) + \hbar l, \tag{12}$$

(μ, ν, l) are independent parameters playing the role of integration constants. They must satisfy the condition $\mu\nu \neq 1$ in order to guarantee that S_0 never takes a constant value. Expression (12) solves (10) without any condition. It is shown in [12] that solution (12) can be written in the form

$$S_0 = \hbar \arctan\left(\mu\frac{\phi_1}{\phi_2} + \nu\right) + \hbar l, \tag{13}$$

with suitable redefining of μ, ν, ϕ_1 and ϕ_2. Another equivalent expression for S_0 was proposed in [6].

In [13] and [14], the extension of this solution to higher dimensions in the separated variable case is investigated and the results are contradictory. The aim of this paper is to provide new insights into the construction of the reduced action in higher dimensions. In Sect. 2, we discuss the link between the reduced action and the wave function. In Sect. 3, we will provide several arguments to show that the usual method which consists in assuming the reduced action as a sum of one variable functions, as in classical mechanics, leads to an incomplete solution. In Sect. 4, we make some remarks about quantum trajectories and microstates. Section 5 is devoted to conclusion.

2 The Reduced Action and the Wave Function

Let us consider for stationary states the separated variable case where the potential takes the following form

$$V(x, y, z) = V_x(x) + V_y(y) + V_z(z). \tag{14}$$

Writing the solution for the SE,

$$-\frac{\hbar^2}{2m}\Delta\phi(x, y, z) + V(x, y, z)\phi(x, y, z) = E\phi(x, y, z), \tag{15}$$

in the form

$$\phi(x, y, z) = \phi_x(x)\phi_y(y)\phi_z(z), \tag{16}$$

we deduce that

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_x}{dx^2}+V_x\phi_x=E_x\phi_x, \tag{17}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_y}{dy^2}+V_y\phi_y=E_y\phi_y, \tag{18}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_z}{dz^2}+V_z\phi_z=E_z\phi_z, \tag{19}$$

where E_x, E_y and E_z are real constants satisfying

$$E_x+E_y+E_z=E. \tag{20}$$

In this section, we will make some comments about [14].

The first comment is just of a pedagogical nature. It concerns the establishment of the above relations (17), (18) and (19). In Sect. 2 of [14], the author considered the above relation (20) as an hypothesis and substituted it in (15) in order to obtain (17), (18) and (19). This manner is wrong because E_x, E_y and E_z follow from the procedure of variable separation and as well-known they are integration constants.

Another comment concerns the form of the three functions $\phi_q(q)$ which are written in Sect. 2 of [14] as

$$\phi_q(q)=R_q(q)\left[\alpha_q\exp\left(i\frac{S_{0q}(q)}{\hbar}\right)+\beta_q\exp\left(-i\frac{S_{0q}(q)}{\hbar}\right)\right], \tag{21}$$

$R_q(q)$ and $S_{0q}(q)$ being real functions, α_q and β_q complex constants and $q=x,y,z$. Substituting (21) in (16), we obtain

$$\begin{aligned}\phi(x,y,z)=R_xR_yR_z\bigg\{&\alpha_x\alpha_y\alpha_z\exp\left[\frac{i}{\hbar}\left(S_{0x}+S_{0y}+S_{0z}\right)\right]\\&+\alpha_x\alpha_y\beta_z\exp\left[\frac{i}{\hbar}\left(S_{0x}+S_{0y}-S_{0z}\right)\right]+\cdots\bigg\},\end{aligned} \tag{22}$$

where the six missing terms can be easily completed. Whatever the expression of the reduced action S_0 in terms of S_{0x}, S_{0y} and S_{0z}, we see that the form (22) cannot reproduce (7).

In addition, it is stated in [14] that the above form (21) is justified in [4, 5]. We emphasize that this is wrong. In [4, 5], the form (21) is established in the context of the equivalence postulate of Faraggi and Matone for the one-dimensional case. In higher dimensions, the relation between the wave function and the couple (R, S_0) is established in the same context in [7] and it is given by the above (7). Admittedly the separated variable case is particular but relation (7) continues to work and never reduces to (22) which follows straightforwardly from (21). Thus, contrary to the statement made in [14], relations (21) and (22) are not in agreement with the equivalence postulate of quantum mechanics [4–7] as they are not with Bohmian mechanics.

The last comment is to say that the procedure used in the construction of the reduced action in [14] leads to a result which does not contain a complete information about the quantum motion. To justify this, some mathematical details are necessary. We come back to this feature in the next section.

We would like to add that all these weakness are reproduced in the others sections of [14] where spherical and cylindrical symmetries are considered and in [15] devoted to the hydrogen atom.

3 The Reduced Action as a Solution of the QHJE

Let us call (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) three couples of real independent solutions respectively of (17), (18) and (19). It follows that the three-dimensional SE admits eight real independent solutions

$$\begin{cases} \phi_1 = X_1 Y_1 Z_1, & \phi_2 = X_1 Y_1 Z_2, \quad \phi_3 = X_1 Y_2 Z_1, \quad \phi_4 = X_1 Y_2 Z_2, \\ \phi_5 = X_2 Y_1 Z_1, & \phi_6 = X_2 Y_1 Z_2, \quad \phi_7 = X_2 Y_2 Z_1, \quad \phi_8 = X_2 Y_2 Z_2. \end{cases} \tag{23}$$

By imposing the invariance of the reduced action, up to an additive constant, under any linear transformation of the solutions of the SE, we showed from the above relations (8) and (9) that the reduced action in the separated variable case is given by [13]

$$S_0^{(1)} = \hbar \text{ arctan}\left(\frac{\sum_{i=1}^{8} \nu_i \phi_i}{\sum_{i=1}^{8} \mu_i \phi_i}\right) + \hbar l, \tag{24}$$

in which we can fix freely one parameter among ν_i and one among μ_i. The fourteen remaining pertinent parameters play the role of integration constants.

In Sect. 2 of [14], the author wrote the reduced action in the following form

$$S_0^{(2)}(x, y, z) = S_{0x}(x) + S_{0y}(y) + S_{0z}(z), \tag{25}$$

as in classical mechanics. In addition, it is assumed in [14] that in this sum S_{0x} is the solution of the usual one-dimensional QHJE, (10), as it is assumed for S_{0y} and S_{0z} with analogous equations. Let us show that the above expression (25) of $S_0^{(2)}(x, y, z)$ does not contain a complete information about the motion of the particle and it is a particular case of (24). For this purpose, by taking into account (12), let us write (25) as

$$\begin{aligned} S_0^{(2)} = {} & \hbar \text{arctan}\left(\frac{X_1 + \gamma_1 X_2}{\gamma_2 X_1 + X_2}\right) + \hbar \text{arctan}\left(\frac{Y_1 + \gamma_3 Y_2}{\gamma_4 Y_1 + Y_2}\right) \\ & + \hbar \text{arctan}\left(\frac{Z_1 + \gamma_5 Z_2}{\gamma_6 Z_1 + Z_2}\right) + \hbar l, \end{aligned} \tag{26}$$

in which $\gamma_1, \ldots, \gamma_6$ are integration constants and $\hbar l$ represents the sum of the three additive constants associated to S_{0x}, S_{0y} and S_{0z}. Let us search for a function h defined by

$$\arctan h = \arctan f_x + \arctan f_y + \arctan f_z, \tag{27}$$

where

$$f_x = \frac{X_1 + \gamma_1 X_2}{\gamma_2 X_1 + X_2}, \tag{28}$$

$$f_y = \frac{Y_1 + \gamma_3 Y_2}{\gamma_4 Y_1 + Y_2}, \tag{29}$$

$$f_z = \frac{Z_1 + \gamma_5 Z_2}{\gamma_6 Z_1 + Z_2}. \tag{30}$$

Knowing that $\tan(a+b) = (\tan a + \tan b)/(1 - \tan a \tan b)$, from (27) we deduce that

$$\begin{aligned} h &= \tan\left[\arctan f_x + \arctan f_y + \arctan f_z\right] \\ &= \frac{\tan(\arctan f_x + \arctan f_y) + f_z}{1 - \tan(\arctan f_x + \arctan f_y) f_z} \\ &= \frac{(f_x + f_y)(1 - f_x f_y)^{-1} + f_z}{1 - (f_x + f_y)(1 - f_x f_y)^{-1} f_z} \\ &= \frac{f_x + f_y + f_z - f_x f_y f_z}{1 - f_x f_y - f_x f_z - f_y f_z}. \end{aligned} \tag{31}$$

Substituting f_x, f_y and f_z by their expressions (28), (29) and (30), we obtain

$$\begin{aligned} h = \Big[& (X_1 + \gamma_1 X_2)(\gamma_4 Y_1 + Y_2)(\gamma_6 Z_1 + Z_2) \\ & + (\gamma_2 X_1 + X_2)(Y_1 + \gamma_3 Y_2)(\gamma_6 Z_1 + Z_2) \\ & + (\gamma_2 X_1 + X_2)(\gamma_4 Y_1 + Y_2)(Z_1 + \gamma_5 Z_2) \\ & - (X_1 + \gamma_1 X_2)(Y_1 + \gamma_3 Y_2)(Z_1 + \gamma_5 Z_2) \Big] \\ & \times \Big[(\gamma_2 X_1 + X_2)(\gamma_4 Y_1 + Y_2)(\gamma_6 Z_1 + Z_2) \\ & - (X_1 + \gamma_1 X_2)(Y_1 + \gamma_3 Y_2)(\gamma_6 Z_1 + Z_2) \\ & - (X_1 + \gamma_1 X_2)(\gamma_4 Y_1 + Y_2)(Z_1 + \gamma_5 Z_2) \\ & - (\gamma_2 X_1 + X_2)(Y_1 + \gamma_3 Y_2)(Z_1 + \gamma_5 Z_2) \Big]^{-1}, \end{aligned} \tag{32}$$

which can be written in the form

$$h = \frac{\sum_{i=1}^{8} \lambda_i \phi_i}{\sum_{i=1}^{8} \delta_i \phi_i}, \tag{33}$$

where $\phi_i (i = 1, \ldots, 8)$ are defined in (23). From (32), the sixteen coefficients λ_i and δ_i can be easily expressed in terms of the six parameters $\gamma_1, \ldots, \gamma_6$. Taking into account relations (27), (28), (29), (30) and (33), expression (26) turns out to be

$$S_0^{(2)} = \hbar \arctan\left(\frac{\sum_{i=1}^{8} \lambda_i \phi_i}{\sum_{i=1}^{8} \delta_i \phi_i}\right) + \hbar l. \tag{34}$$

Comparing (24) and (34), we see clearly that $S_0^{(1)}$ and $S_0^{(2)}$ have the same form. However, in expression (24) there are fourteen independent parameters among $(\nu_1, \ldots, \nu_8)$ and $(\mu_1, \ldots, \mu_8)$ while in (34) there are six independent parameters $\gamma_1, \ldots, \gamma_6$. Thus, solution (34) can be obtained by choosing particular values for eight $(8 = 14 - 6)$ parameters in (24). This is the proof that the solution (25) is a particular case of (24).

Furthermore, we can show that it is (34) which is lacking in parameters and not in (24) that there is a surplus. In fact, even by supposing that (25) is true, we will show that (8) and (9) lead to a solution more general than (26). For this purpose, observe that expression (16) for the wave function indicates that

$$R(x,y,z) = R_x(x)R_y(y)R_z(z). \tag{35}$$

This last expression is suggested in [7] and implicitly assumed in [14]. So, by setting $S_0(x,y,z)$ as in (25) and $R(x,y,z)$ as in (35), we obtain from (8)

$$\begin{aligned}&\frac{1}{2m}\left(\frac{\partial S_{0x}}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial S_{0y}}{\partial y}\right)^2 + \frac{1}{2m}\left(\frac{\partial S_{0z}}{\partial z}\right)^2 - \frac{\hbar^2}{2mR_x}\frac{\partial^2 R_x}{\partial x^2}\\ &\quad -\frac{\hbar^2}{2mR_y}\frac{\partial^2 R_y}{\partial y^2} - \frac{\hbar^2}{2mR_z}\frac{\partial^2 R_z}{\partial z^2} + V_x(x) + V_y(y) + V_z(z) = E,\end{aligned} \tag{36}$$

where we have used (14). The procedure of variable separation leads to the three following equations

$$\frac{1}{2m}\left(\frac{\partial S_{0x}}{\partial x}\right)^2 - \frac{\hbar^2}{2mR_x}\frac{\partial^2 R_x}{\partial x^2} + V_x(x) = E_x, \tag{37}$$

$$\frac{1}{2m}\left(\frac{\partial S_{0y}}{\partial y}\right)^2 - \frac{\hbar^2}{2mR_y}\frac{\partial^2 R_y}{\partial y^2} + V_y(y) = E_y, \tag{38}$$

$$\frac{1}{2m}\left(\frac{\partial S_{0z}}{\partial z}\right)^2 - \frac{\hbar^2}{2mR_z}\frac{\partial^2 R_z}{\partial z^2} + V_z(z) = E_z. \tag{39}$$

The three integration constants E_x, E_y and E_z satisfy the condition (20). Substituting expressions (25) and (35) in (9) and dividing then the obtained relation by R^2, we find

$$\frac{1}{R_x^2}\frac{\partial}{\partial x}\left(R_x^2\frac{\partial S_{0x}}{\partial x}\right) + \frac{1}{R_y^2}\frac{\partial}{\partial y}\left(R_y^2\frac{\partial S_{0y}}{\partial y}\right) + \frac{1}{R_z^2}\frac{\partial}{\partial z}\left(R_x^2\frac{\partial S_{0z}}{\partial z}\right) = 0. \tag{40}$$

From this relation, the procedure of variable separation leads to

$$\frac{1}{R_x^2}\frac{\partial}{\partial x}\left(R_x^2\frac{\partial S_{0x}}{\partial x}\right) = c_1, \tag{41}$$

$$\frac{1}{R_y^2}\frac{\partial}{\partial y}\left(R_y^2\frac{\partial S_{0y}}{\partial y}\right) = c_2, \tag{42}$$

$$\frac{1}{R_z^2}\frac{\partial}{\partial z}\left(R_x^2\frac{\partial S_{0z}}{\partial z}\right) = c_3. \tag{43}$$

Taking into account (40), the integration constants c_1, c_2 and c_3 must satisfy the condition

$$c_1 + c_2 + c_3 = 0. \tag{44}$$

With the use of x as variable, the usual one-dimensional case can be obtained from (41) by setting $c_1 = 0$. The presence of the constants c_1, c_2 and c_3 in (41), (42) and (43) is the reason for which the extension to three dimensions is not similar to classical mechanics. In fact, rewriting (41) in the form

$$\frac{2}{R_x}\frac{\partial R_x}{\partial x}\frac{\partial S_{0x}}{\partial x} + \frac{\partial^2 S_{0x}}{\partial x^2} = c_1, \tag{45}$$

and taking the derivative with respect to x, we find

$$\frac{2}{R_x}\frac{\partial^2 R_x}{\partial x^2}\frac{\partial S_{0x}}{\partial x} - \frac{2}{R_x^2}\left(\frac{\partial R_x}{\partial x}\right)^2\frac{\partial S_{0x}}{\partial x} + \frac{2}{R_x}\frac{\partial R_x}{\partial x}\frac{\partial^2 S_{0x}}{\partial x^2} + \frac{\partial^3 S_{0x}}{\partial x^3} = 0. \tag{46}$$

From (45), we have

$$\frac{\partial R_x}{\partial x} = -\frac{R_x}{2}\left(\frac{\partial S_{0x}}{\partial x}\right)^{-1}\left(\frac{\partial^2 S_{0x}}{\partial x^2} - c_1\right). \tag{47}$$

Substituting this expression in (46), we find

$$\begin{aligned}\frac{1}{R_x}\frac{\partial^2 R_x}{\partial x^2} &= -\frac{1}{2}\left(\frac{\partial S_{0x}}{\partial x}\right)^{-1}\frac{\partial^3 S_{0x}}{\partial x^3}\\ &\quad + \frac{1}{4}\left(\frac{\partial S_{0x}}{\partial x}\right)^{-2}\left[3\left(\frac{\partial^2 S_{0x}}{\partial x^2}\right)^2 - 4c_1\frac{\partial^2 S_{0x}}{\partial x^2} + c_1^2\right].\end{aligned} \tag{48}$$

Using this result in (37), we obtain

$$\begin{aligned}&\frac{1}{2m}\left(\frac{\partial S_{0x}}{\partial x}\right)^2 - \frac{\hbar^2}{4m}\left[\frac{3}{2}\left(\frac{\partial S_{0x}}{\partial x}\right)^{-2}\left(\frac{\partial^2 S_{0x}}{\partial x^2}\right)^2 - \left(\frac{\partial S_{0x}}{\partial x}\right)^{-1}\frac{\partial^3 S_{0x}}{\partial x^3}\right]\\ &\quad + V_x(x) - E_x = \frac{\hbar^2 c_1}{8m}\left(\frac{\partial S_{0x}}{\partial x}\right)^{-2}\left[c_1 - 4\frac{\partial^2 S_{0x}}{\partial x^2}\right].\end{aligned} \tag{49}$$

In the same manner, we can also obtain

$$\begin{aligned}&\frac{1}{2m}\left(\frac{\partial S_{0y}}{\partial y}\right)^2 - \frac{\hbar^2}{4m}\left[\frac{3}{2}\left(\frac{\partial S_{0y}}{\partial y}\right)^{-2}\left(\frac{\partial^2 S_{0y}}{\partial y^2}\right)^2 - \left(\frac{\partial S_{0y}}{\partial y}\right)^{-1}\frac{\partial^3 S_{0y}}{\partial y^3}\right]\\ &\quad + V_y(y) - E_y = \frac{\hbar^2 c_2}{8m}\left(\frac{\partial S_{0y}}{\partial y}\right)^{-2}\left[c_2 - 4\frac{\partial^2 S_{0y}}{\partial y^2}\right],\end{aligned} \tag{50}$$

$$\begin{aligned}&\frac{1}{2m}\left(\frac{\partial S_{0z}}{\partial z}\right)^2 - \frac{\hbar^2}{4m}\left[\frac{3}{2}\left(\frac{\partial S_{0z}}{\partial z}\right)^{-2}\left(\frac{\partial^2 S_{0z}}{\partial z^2}\right)^2 - \left(\frac{\partial S_{0z}}{\partial z}\right)^{-1}\frac{\partial^3 S_{0z}}{\partial z^3}\right]\\ &\quad + V_z(z) - E_z = \frac{\hbar^2 c_3}{8m}\left(\frac{\partial S_{0z}}{\partial z}\right)^{-2}\left[c_3 - 4\frac{\partial^2 S_{0z}}{\partial z^2}\right].\end{aligned} \tag{51}$$

In the left hand side of (49), we recognize the usual one-dimensional QHJE. Since for an arbitrary potential $V_x(x)$, $c_1 - 4\partial^2 S_{0x}/\partial x^2 \neq 0$, the usual one-dimensional QHJE is a particular case of (49) which happens only if $c_1 = 0$. Therefore, the decomposition (25) and (35) does not automatically reproduce the usual one-dimensional case. This means that the procedure consisting to search for the reduced action in the form (25) as suggested in [14] spreads confusion. This follows from a profound geometrical origin. In fact, it is shown in [7] that assuming the decomposition (25) and (35) leads to express the quantum potential as a sum of Schwarzian derivatives and does not provide a covariant formulation within the framework of the equivalence postulate.

In conclusion, even with the assumption (25), the presence of the constants c_1, c_2 and c_3 in relations (49), (50) and (51) means that in the above expression (26) of $S_0^{(2)}$ there is a lack of integration constants. So, in $S_0^{(2)}$ as suggested in [14], there is a loss of information concerning the quantum motion.

We would like to add that the above solution (24) established in [13] is really complete. In fact, from the mathematical point of view, the SE is strictly equivalent to the couple of (8) and (9) in the case where the wave function is complex. In this case, it is established in [13] that by fixing the initial conditions for the most general solution of the SE, the above expression (24) contains the exact number of pertinent parameters necessary to fix univocally the reduced action, as it is well-known in one dimension. This represents a strong argument in favour of (24). In the next section, we will come back to this problem of initial conditions.

4 The quantum trajectories

Trajectories in the context of quantum mechanics were introduced by Einstein [16, 17], de Broglie [1] and Bohm [2, 3]. For various reasons described in the literature [12, 18–20], other formulations were published in the one-dimensional stationary case. In [12], by appealing to the quantum transformation [6, 23] allowing to write the QHJE in the classical form, the relation

$$\frac{1}{2}\frac{\partial S_0}{\partial x}\dot{x} + V(x) = E, \tag{52}$$

is derived. This relation has allowed to establish the quantum Newton law. Although (52) works in classical mechanics ($\partial S_0^{clas}/\partial x = m\dot{x}$), it describes the quantum motion because $\partial S_0/\partial x$ is the solution of the QHJE (10). The higher dimension version of (52) can be sensibly assumed as

$$\frac{1}{2}\frac{\partial S_0}{\partial x}\dot{x} + \frac{1}{2}\frac{\partial S_0}{\partial y}\dot{y} + \frac{1}{2}\frac{\partial S_0}{\partial z}\dot{z} + V(x, y, z) = E. \tag{53}$$

What is at stake is how to prove this relation. In [21], the author claimed that he provided a proof. However, his reasoning is false. In fact, it is obvious that the identification of (56) and (58) in [21] is erroneous because the author compared in a development with respect to derivatives of S_0 coefficients which are themselves depending on these derivatives. In addition, the author used the wrong expressions of these coefficients to determine trajectories in the three-dimensional constant potential case while the above law (53) should not be enough to describe a motion in higher dimensions. Even in classical mechanics, it is known that the use of the energy conservation equation is insufficient to describe a motion except in one dimension.

We would like to indicate that a correct proof of (53) is provided in [22] in which we stressed that this law must be completed in order to be applied in the case where the dimension of space is more than one.

The last point that we would like to raise concerns microstates associated to various trajectories for the same physical state. In any dimension, if the variables are separable, the manifestation of microstates is occurred in the case where the system is described by a real wave function, up to a constant phase factor. In the case of complex wave function, it is established in [13] that by fixing the initial conditions for the most general solution of the three-dimensional SE,

$$\phi(x, y, z) = \sum_{i=1}^{8} c_i \phi_i, \tag{54}$$

the reduced action is univocally fixed, (c_i) being a set of eight complex constants and (ϕ_i) the real solutions of the SE defined in (23). In others words, there is no microstates. However, in the context of the Copenhagen interpretation and separated variables, the probability density in the multidimensional configuration space is the product of individual probability densities. This is the reason for which the wave function is often sought in the form given by the above relation (16) in which $\phi_x(x)$, $\phi_y(y)$ and $\phi_z(z)$ are the general solutions

$$\begin{aligned} &\phi_x(x) = a_x X_1 + b_x X_2, \qquad \phi_y(y) = a_y Y_1 + b_y Y_2, \\ &\phi_z(z) = a_z Z_1 + b_z Z_2 \end{aligned} \tag{55}$$

of the corresponding one-dimensional SEs. In (55), X_1, X_2, Y_1, Y_2, Z_1 and Z_2 are the same functions as in (23) and $(a_x, b_x, a_y, b_y, a_z, b_z)$ is a set of complex constants. Substituting the one-dimensional general solutions (55) in (16), we obtain the same form as in (54)

$$\phi(x, y, z) = \sum_{i=1}^{8} c'_i \phi_i . \tag{56}$$

Contrary to (54) in which the eight coefficients c_i are independent, in (56) the c'_i can be expressed in terms of the six parameters a_x, b_x, a_y, b_y, a_z and b_z. To be more precise, a simple rearrangement shows that c'_i can be expressed in terms of only four independent parameters. Thus, fixing the initial conditions for the wave function and following the same procedure as the one developed in [13], we cannot determine all the integration constants in (24) and then we do not fix completely the reduced action. This reveals the presence of microstates. Contrary to the one-dimensional case [9, 24], we conclude that the QHJE describes microstates not detected by the SE even in the case of a complex wave function if it is constructed from (16).

5 Conclusion

To summarize, we pointed out in this paper some flaws in earlier publications concerning in higher dimensions the construction of the reduced action as a solution of the QHJE [14, 15] and the establishment of the quantum law of motion [21]. We also indicated how to correct these flaws and provided new insights into the investigation of the QHJE. In particular, we proved that the reduced action constructed as a sum of one variable functions, as in classical mechanics, does not contain a complete information about the quantum motion. Finally, we established that in higher dimensions the QHJE describes microstates not detected by the SE even in the case where the wave function is complex. From the mathematical point of view, we stress to indicate that in the case where the wave function is complex, the QHJE is equivalent to the SE. In fact, if the wave function is written as in (54) where the eight coefficients c_i are independent, there is no trace of microstates [13]. However, the existence of various trajectories for the same state described by a complex wave function is a consequence of the relationships between the eight coefficients c'_i appearing in (56) and imposed by the Copenhagen interpretation.

References

1. de Broglie, L., Ann. Fond. Louis Broglie **12**, 1 (1987)

2. Bohm, D.: Phys. Rev. **85**, 166 (1952)
3. Bohm, D.: Phys. Rev. **85**, 180 (1952)
4. Faraggi, A.E., Matone, M.: Phys. Lett. B **450**, 34 (1999)
5. Faraggi, A.E., Matone, M.: Phys. Lett. B **437**, 369 (1998)
6. Faraggi, A.E., Matone, M.: Int. J. Mod. Phys. A **15**, 1869 (2000)
7. Bertoldi, G., Faraggi, A.E., Matone, M.: Class. Quantum Gravity **17**, 3965 (2000)
8. Poirier, B.: J. Chem. Phys. **121**, 4501 (2004)
9. Bouda, A.: Found. Phys. Lett. **14**, 17 (2001)
10. Floyd, E.R.: Phys. Rev. D **34**, 3246 (1986)
11. Floyd, E.R.: Phys. Essays **5**, 130 (1992)
12. Bouda, A., Djama, T.: Phys. Lett. A **285**, 27 (2001)
13. Bouda, A., Mohamed Meziane, A.: Int. J. Theor. Phys. **45**, 1323 (2006)
14. Djama, T.: Phys. Scr. **75**, 77 (2007)
15. Djama, T.: arXiv:quant-ph/0404175 (2004)
16. Holland, P.: Found. Phys. **35**, 177 (2005)
17. Belousek, D.W.: Stud. His. Philos. Mod. Phys. **27**, 437 (1996)
18. Floyd, E.R.: Phys. Rev. D **26**, 1339 (1982)
19. Bouda, A.: Int. J. Mod. Phys. A **18**, 3347 (2003)
20. Wyatt, R.E.: Quantum Dynamic with Trajectories. Springer, Berlin (2005). ISBN:0-387-22964-7
21. Djama, T.: Phys. Scr. **76**, 82 (2007)
22. Bouda, A., Gharbi, A.: Int. J. Theor. Phys. **47**, 1068 (2008)
23. Faraggi, A.E., Matone, M.: Phys. Lett. A **249**, 180 (1998)
24. Floyd, E.R.: Found. Phys. Lett. **9**, 489 (1996)